MODULE 11: SELECTED TOPICS IN NON-CONSTANT RATES

Life would be much simpler if all rates were constant, but the fact is that the most important rates are not constant. We learn at different rates, we grow at different rates, and we earn at different rates. Even within ourselves rates are not constant. When we're twice as old, we're not twice as smart, nor twice as tall, nor twice as rich.

In this module we want to look at some common non-constant rates.

Category 1: Mixtures of Different Constant Rates

In Module 9 we talked about a merchant buying a certain number of coats at a certain constant price per coat. In this case we found the total expense of the coats by multiplying the number of coats by the price per coat. *But what if there were different prices for different coats?*

Example 1

> A merchant buys 30 coats at $40 per coat and 70 coats at $60 per coat. What was the total cost of the 100 coats?

Answer: $5,400

Here we have two constant rate problems.

At $40 per coat, 30 coats will cost $40, thirty times; that is:

$40 X 30 or $1,200

This by itself could have been a Module 9 example.

At $60 per coat, 70 coats will cost

$60 X 70 or $4,200

This by itself could have been a Module 9 example.

We then add $1,200 and $4,200 to get $5,400 as the answer.

The two results combined become a Module 11 example.

When we have more than one rate, we often talk about an *average* rate. For example, in Example 1 we see that 100 coats cost \$5,400. To find the average cost per coat, we divide the total cost (\$5,400) by the total number of coats (100). That is:

$$\text{Average cost (In \$ per coat)} = \frac{\text{Total cost in dollars}}{\text{Total number of coats}} = \frac{\$5{,}400}{100} = \$54$$

If the coats were equally-priced then the average cost per coat would have been the same as the actual cost per coat.

Notice that the average price is not a good indicator of the actual prices. The only two prices the merchant paid were \$40 and \$60. Not a single coat cost \$54. In fact, any combination of purchases in which the merchant bought 100 coats for \$5,400 would have resulted in an average price of \$54 per coat.

One of the main topics of the subject called "statistics" is to study how the average cost is related to the actual individual costs.

Example 2

A merchant buys 20 sweaters at \$20 per sweater; 10 sweaters at \$40 per sweater; and 50 sweaters at \$60 per sweater.

(a) What was the total cost for the sweaters?

(b) What was the average cost per sweater?

Answers: (a) \$3,800
(b) \$47.50

Here we're dealing with three different constant rate problems.

(1) We have 20 sweaters @ \$20. So that cost is \$400.

(2) We have 10 sweaters @ \$40. So that cost is \$400

(3) We have 50 sweaters @ \$60. So that Cost is \$3,000

@ is a common business abreviation that stands for the cost per item.

20 X \$20 = \$400
10 X \$40 = \$400
50 X \$60 = \$3,000

This accounts for the entire mixture of sweaters. Hence:

(a) We find the total cost by adding the amounts we found in (1), (2), and (3). This gives us \$400 + \$400 + \$3,000 or \$3,800

In tabluated form we usuall write:

20 sweaters @ \$20	*=*	*\$ 400*
10 sweaters @ \$10	*=*	*400*
50 sweaters @ \$50	*=*	*3,000*
80 sweaters	*=*	*\$3,800*

So had the sweaters been equally-priced (which they weren't) then the actual price per sweater would hav been \$47.50

(b) And to find the average price per sweater, we take the total cost (\$3,800) and divide it by the total number of sweaters (20 + 10 + 50 = 80)to get: \$3,800 ÷ 80 or \$47.50

<u>Check:</u>

80 sweaters @\$47.50 cost 80 X \$47.50 and this comes out to be \$3,800. In other words, any combination of 80 sweaters for \$3,800 would have yielded an average cost of \$47.50 per sweater.

Notice that \$40 is mid-way between \$20 and \$60, but th average cost is more than \$40. The reason is that th majority of the sweaters (50 of the 80) cost \$60.

There is no limit to how many different rates can be contained in one problem; nor is there a limit as to how many of each rate are present. *An interesting, yet common, special case is when there is only one of each rate present.* This happens, for example, when you pay your bills by check. Each check is probably for a different amount. Hence the amount per check is a rate that's different for each check. Or when you take tests, you usually get different grades. In this context, the points per test is usually different for each test.

We assume, of course that there is at least one of a given rate present, otherwise we don't need that rate

Remember that a rate is involved whenever "per" is between two nouns, as in "amount per check" or "points per test"

Most students are very much interested in their "average when it comes to points per test.

Let's look at a few examples.

Example 3

> In 5 tests you get 85 points each.
> How many points did you get as a total
> for these five tests?

Answer: 425

Since you got 85 points per test and there were 5 tests, you scored 85 points five times for a total of 85 X 5 or 425 points.

In Example 3, because the rate of points per test was constant, the 85 also represents your average score. Usually, however, your point-scoring rate will not be constant.

Example 4

> You take four tests and receive scores of 80, 82, 83, and 88. How many points must you get on the next test in order to have an average of 85 points per test?

Answer: 92

Based on what we did in Example 3, it is easy to see that we need 425 total points in order that the average be 85 points per test.

So we begin by seeing how many points we already have. Namely:

$$80 + 82 + 83 + 88 = 333$$

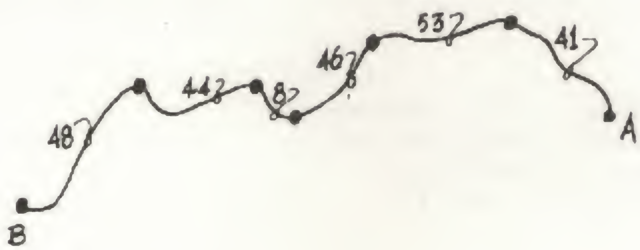
If you're using a calculator writing the summands on the same line is convenient. If you're adding the "traditional" way, it is easy to align the numbers vertically.

We next determine how much has to be added to 333 to get 425. That is, we subtract 333 from 425 to get 425 - 333 = 92.

Notice how in this one problem we had to be able to multiply (85 X 5), add (80 + 82 + 83 + 88), and subtract (425 - 333).

See what happens here? The 80 was 5 less than 85, 82 was 3 less, and 83 was 2 less So that puts us 5 + 3 + 2 or 10 points below an 85 average. The 88 makes up 3 of these 10. So you still have to make up 7. 85 + 7 = 92

Computing the average cost of a bunch of sweaters or the average score of a bunch of tests may seem highly specialized, but the type of reasoning involved comes up in many common experiences. For example, have you ever had to use a road map to help plan a trip? Look at Figure 1. It gives us the mileage from point A to point B, but in a form that's piece by piece.



See the non-constant rate the map gives us? Each number represents the length (in miles) per given segment.

Figure 1

<u>Example 5</u>

Based on Figure 1, how many miles is it from point A to point B?

Answer: 240 miles

We start at point A and add the lengths of each segment that takes us from A to B. We get:

$$\begin{array}{rl} & 41 \text{ miles} \\ + & 53 \text{ miles} \\ + & 46 \text{ miles} \\ + & 8 \text{ miles} \\ + & 44 \text{ miles} \\ + & \underline{48 \text{ miles}} \\ & 240 \text{ miles} \end{array}$$

Once we know the distance, other rates can become important.

Example 6

With A and B as in Example 5, what must your average speed be if you want to make the trip in 6 hours?

Answer: 40 miles per hour

The average speed is obtained by dividing the distance you have to go by the time it takes to travel that distance. So in this case, we have:

240 miles ÷ 6 hours = 40 miles per hour

From another point of view, if you know that you have to make the trip in less than 6 hours, then you also know that you'll have to average more than 40 mph

Example 7

Again referring to Examples 5 and 6, suppose you know that for this kind of trip your car will give you 15 miles per gallon of gas. How many gallons of gas will you use for the trip?

Answer: 16 gallons

Using the approach of Module 9, we have:

$$\frac{240 \text{ miles}}{1} \times \frac{1 \text{ gallon}}{15 \text{ miles}} =$$

$$\frac{240 \cancel{\text{miles}} \times 1 \text{ gallon}}{1 \times 15 \cancel{\text{miles}}} =$$

$$\frac{240 \text{ gallons}}{15} =$$

$$\frac{240}{15} \text{ gallons or } 16 \text{ gallons}$$

In effect we're taking the distance (240 miles) and dividing it by 15 mpg. The fact that we're dividing is seen from the fact that 15 is in the denominator.

Example 8

Under the conditions of Examples 5, 6, and 7, suppose you pay $1.19 per gallon of gas. How much will you have to pay for the gas you use on the trip?

Answer: $19.04

Every gallon of gas will cost us $1.19, and we need 16 gallons. Hence we have to pay $1.19, sixteen times--or:

$1.19 X 16 = $19.04

Of course we don't expect that $19.04 will be the exact cost, but it gives us a good estimate. For example, if we allow $25 for gas we should be on the safe side.

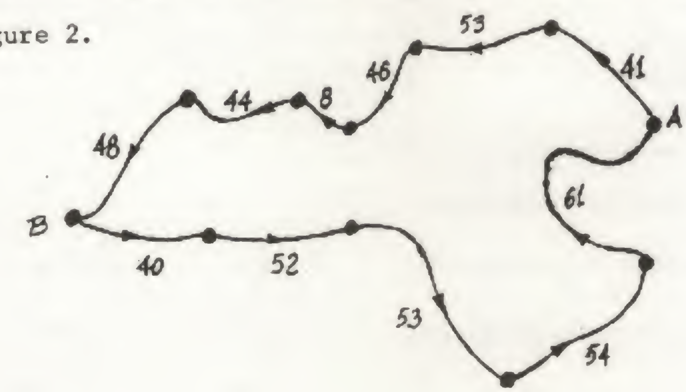
Hopefully Examples 5, 6, 7, and 8 give you a good idea of how often we use rates in making decisions. Additional drill will be left for the Self-Test.

We often say such things as "At this rate, we'll never finish on time"; and this might lead to the decision that we must work faster.

Example 5 introduces us to another category of using rates. Namely:

Category 2: Perimeter, Area, and Volume

Sometimes we take a trip that forms a *closed path*; that is, a trip that takes us back eventually to the same point from which we started. With respect to Example 5, suppose that we decided to return to A after we got to B. We might or might not take the same path. One possibility is depicted in Figure 2.



On sightseeing trips we often take a different path returning than we took going so that we can see as many different things as possible.

Figure 2

Example 9

> Based on Figure 2, if you use the path shown what was your total mileage in making the round trip between A and B?

Answer: 500 miles

We already know that the length of the upper path is 240 miles (from Example 5). So all we have to do now is find the length of the lower path and add this to the length

Because addition is both commutative and associative it makes no difference whether we measure from A to B or from B to A.

of the upper path. We get:

```
    40 miles
 +  52 miles
 +  53 miles
 +  54 miles
 +  61 miles
   260 miles (= length of lower path)
 + 240 miles (= length of upper path)
   500 miles (= length of round trip)
```

Notice that the round trip path in Figure 2 encloses a region that we've denoted by the letter R. With respect to the region R we've gone around its boundary. *We give a special name to the length of the boundary that encloses a region.*

R is called a closed region because its boundary is a closed curve. That is, its boundary begins and ends at the same point (either A or B, or, for that matter, any point on the boundary)

> *Definition*
>
> *Given a closed region R, the* perimeter *of R is the length of the boundary that encloses R.*

So with respect to Figure 2, the perimeter of R is 500 miles.

The easiest closed regions for finding perimeters are those whose boundaries are straight line segments.

By a "segment" we mean "part" of a straight line. In most geometry courses, a straight line is assumed to go on "forever" (indefinitely). Hence if we "chop off" the line we call it a line segment.

> *Definition*
>
> *A closed region is called* rectilinear *if its boundaries are made up of straight line segments.*

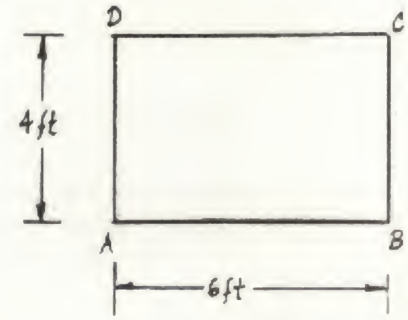
A very useful rectilinear region is a *rectangle.*

The most common non-rectilinear region is called a circle. Circles are discussed in Appendix 2.

> *Definition*
>
> *A four-sided rectilinear region with "square" corners is called a* rectangle.
>
> *The side the rectangle "rests on" is called the* base *and the side next to the base is called the* height.

We sometimes use the words "length" and "width" in place of "base" and "height"

Let's make sure we understand the new vocabulary.



Example 10

Referring to Figure 3 in the margin:

(a) What is the length of the base of the rectange?

(b) How long is the height of the rectangle?

(c) What is the perimeter of the rectangle?

Answers: (a) 6 feet
(b) 4 feet
(c) 20 feet

(a) The base is the side the rectangle rests on. In Figure 3, we see that the length of this side is 6 feet.

(b) The side next to the base is called the height. The length of this side is 4 feet.

(c) The perimeter is the total length of the boundary of the rectangle. The total length of the base and the height of the rectangle is 6 feet + 4 feet or 10 feet. Since the length of the base appears twice in the boundary, as does the length of the height, we see that the perimeter is 2 X 10 feet or 20 feet.

This is the key point about the perimeter of a rectangle. The side opposite the base has the same length as the base; and the length of the side opposite the height has the same length as the height.

To Find the Perimeter of a Rectangle

Step 1: Multiply the (length of the) base by 2.

Step 2: Multiply the (length of the) height by 2.

Step 3: Add the two answers in Steps 1 and 2.

By the distributive property we could first add the base and the height and then multiply by 2.

Some Colloquiol Language

If the base of a rectangle is 6 feet long and the height is 4 feet long we often refer to the rectangle as a "6 foot by 4 foot" rectangle.

The shape and size of the rectangle do not change if we make the 4 foot side the base and the 6 foot side the height.

Example 11

What is the perimeter of a 9 foot by 1 foot rectangle?

Answer: 20 feet

The rectangle is drawn as Figure 4 in the margin. As drawn, the length of the base is 9 feet and the length of the height is 1 foot. Hence to find the perimeter:

1. We multiply the base by 2 to get

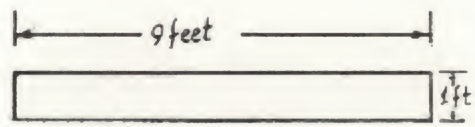
   2 X 9 ft or 18 feet

2. We multiply the height by 2 to get

   2 X 1 foot or 2 feet

3. Then we add the results of the first two steps to get:

   18 feet + 2 feet or 20 feet

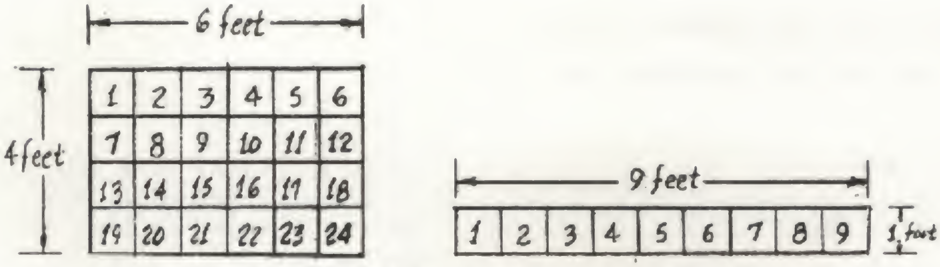


(Figure 4)

The lengths used in Figures 3 and 4 also use constant rates. We did not draw the actual lengths. Rather we used a scale of ¼ inch to stand for each foot. That is, our scale was at a rate of ¼ inch per foot.

Comparing the results of Examples 10 and 11 leads to a very interesting situation. Both rectangles have the same perimeter (20 feet), yet they enclose different amounts of space. That is, the regions enclosed by the two rectangles have different sizes. More specifically, as shown in Figure 5, 24 one-foot-squares fit in the 6 foot by 4 foot rectangle, while only 9 of these squares fit in the 9 foot by 1 foot rectangle.

By a one-foot-square we mean a rectangle, each of whose four sides is 1 foot long. In general, if all four sides of a rectangle have the same length, we call the rectangle a square.



(Figure 5)

See the subtlety? Both rectangles have a perimeter of 20 feet but the two rectangles have a different size. In terms of a more practical example, suppose the rectangles were the shape of a floor. It would take 24 one-foot tiles to make a covering for the firs floor, but only 9 one-foot tiles to cover the second floor.

Notice that we don't have to count the one-foot squares to find the amount of space in each rectangle. For example if the rectangle is 6 feet by 4 feet we have 6 one-foot vertical strips and 4 one-foot horizontal strips. So all in all we have 6 X 4 or 24 one-foot squares.

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<u>Some More Vocabulary</u>

The amount of space enclosed by a rectangle is called the <u>area</u> of the rectangle.

To find the area of a rectangle we multiply the base by the height, being careful to use the same denominations for both measurements.

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<u>Example 12</u>

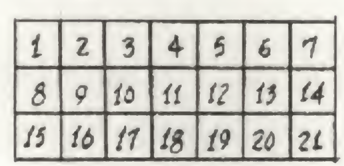
What is the area of a 7 foot by 3 foot rectangle?

Answer: 21 <u>square feet</u>

An area is never labeled "feet". "feet" measures length. "Square feet" measure the space inside the rectangle.

We have 7 X 3 or 21 one-foot squares inside the rectangle. If we call the "amount of space" inside each of the 1 foot squares " a square foot", then we have 21 square feet inside the rectangle.

This is summarized in Figure 6 in the margin



(Figure 6)

Notice how we handle the denominations. When we multiply the base by the height we have:

7 feet X 3 feet =

(7 X 3) feet X feet

Just as we call the product of a number with itself, the square of the number, we call the product of feet and feet "square feet".

7 feet X 3 is 21 feet and 7 X 3 feet is 21 feet; but 7 feet X 3 feet requires that we multiply "feet" by "feet".

We must be careful of denominations.

Example 13

What is the area of a rectangle that is 7 yards by 3 yards?

Answer: 21 square yards

Don't just write "21" as the answer. The label is very important. For example, it makes a difference whether we have 21 square feet or 21 square yards.

This problem is done in exactly the same way as Example 12, except that we deal with yards instead of feet.

This time the area is

7 yards X 3 yards =

(7 X 3) yards X yards =

21 square yards.

If the denominations are different when we measure the base and height, we must switch to a common denomination before we find the area.

Example 14

What is the area of a rectangle that is 7 feet by 3 yards? Write the answer in square feet.

Answer: 63 square feet

To get "square feet" for an area we must multiply "feet" by "feet". With this as a hint we convert 3 yards to feet.

So the area is:

7 feet X 3 yards =

7 feet X (3 X 1 yard) =

7 feet X (3 X 3 feet) =

7 feet X 9 feet =

(7 X 9) feet X feet =

63 square feet

At this stage the answer would be 21 feetyards (?). It is neither feet X feet nor yards X yards

We're just converting 3 yar to 9 feet

Scientists often abbreviate square feet as ft^2 to reflect the fact that we took ft X ft to get square feet

The square is a very special rectangle. For reasons that aren't important to discuss here it turns out that for a given perimeter, the rectangle of greatest area is the square. By way of review, notice that the 6 foot by 4 foot, the 9 foot by 1 foot, and the 7 foot by 3 foot rectangles all have a perimeter of 20 feet. Their areas are ,respectively, 24 square feet, 9 square feet, and 21 square feet.

If the 20 feet were used to form a square, each side would be 5 feet. The area of a 5 foot by 5 foot rectangle (square) is 25 square feet--which is bigger than either 24 square feet, 9 square feet, or 21 square feet.

That is, since all 4 sides have the same length, the length of each side must be ¼ of the perimeter.

Example 15

The length of each side of a square is 8 inches. What is:

(a) the perimeter of the square?

(b) the area of the square?

Answers: (a) 32 inches
(b) 64 square - inches

(a) A square is still a rectangle. The only difference is that for the square the

base and height have the same length. So twice the base is 16 inches and twice the height is 16 inches. So the perimeter is 16 + 16 or 32 inches.

In the case of a square it is quicker to find the perimeter by multiplying the length of any side by 4.

(b) The area is the product of the base and the height, so we get:

8 inches X 8 inches =

(8 X 8) inches X inches =

64 square inches

Try to check for yourself that if the perimeter of a rectangle is 32 inches, its area can't be more than 64 square inches. For example, the perimeter of a 7 inch by 9 inch rectangle is 32 inches, but its area is 63 square inches.

Pictorially:

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

The distance around the square is 32 inches. But the amount of space inside the square is 64 square inches.

See how the labels help? "inches" indicate length; "square inches" indicate area.

To Find the Area of A Square:

(1) Take the length of a side.

(2) "Square" it. That is, multiply it by itself.

Any side will do because for a square all sides have the same length.

Notice that the rate of change of the area of a square with respect to the length of the side of a square is *not constant*.

Example 16

How many square inches are there in:

(a) a 3-inch square?
(b) a 4-inch square?
(c) a 5-inch square?

Answers: (a) 9 (b) 16 (c) 25.

In all 3 parts, we simply multiply the length of a side by itself.

(a) 3 inches X 3 inches =

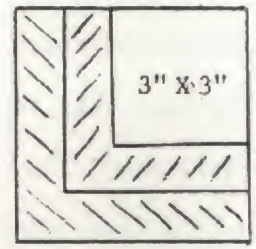
9 square inches

(b) 4 inches X 4 inches =

16 square inches

(c) 5 inches X 5 inches =

25 square inches.



Look at the above square. Everytime the length of a side increases by 1 inch, the area increases by the amount in the L-shaped region.

In each part of Example 16 the length of the side of the square changed by 1 inch, but the changes in the area were 7 square inches $(16\ \text{in}^2 - 9\ \text{in}^2)$ and 9 square inches (25 square inches - 16 square inches).

Example 17

There are 3 feet per yard. How many square feet are there in 2 square yards?

Answer: 18 (not 6!)

Let's let the adjectives and the nouns do the work for us!

1 square yard = 1 yard X 1 yard

= 3 feet X 3 feet

= 9 square feet

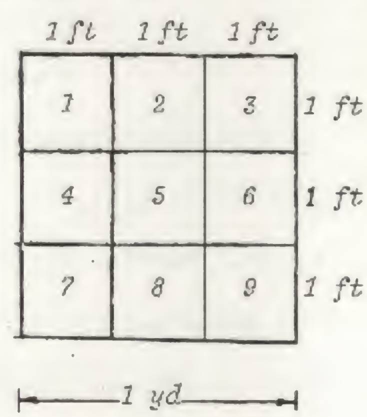
So:

2 square yards = 2 X 1 square yard

= 2 X 9 square feet

= 18 square feet

The fact that there are 3 feet per yard means that there are 9 square feet per square yard. Pictorially:



Example 17 warns us of a very common trap that people sometimes fall into when they deal with area. The fact that there were 3 feet in a yard means that there are 3 X 3 or 9 square feet in a square yard. Notice that we *squared* 3 to get the answer. This procedure is always correct. For example:

Remember that squaring a number means that we multiply the number by itself.

1 square foot = 1 foot X 1 foot
= 12 inches X 12 inches
= 12^2 or 144 square inches

1 square meter = 1 meter X 1 meter
= 100 cms X 100 cms
= 100^2 or 10,000 sq cms.

Example 18

The area of a certain square is 1,296 square inches. What is the area of the square in square feet?

Answer: 9 (square feet)

As shown above, there are 144 square inches per square foot. Hence:

That is, each 144 square inches equal 1 square foot. So we must divide the number of square *inches by 144--* not 12--*to get the number of square feet.*

$$\frac{1{,}296 \text{ square inches}}{1} \times \frac{1 \text{ square foot}}{144 \text{ square inches}} =$$

$$\frac{1{,}296 \cancel{\text{square inches}} \times 1 \text{ square foot}}{1 \times 144 \cancel{\text{square inches}}} =$$

$$\frac{1{,}296}{144} \text{ square feet} =$$

9 square feet

Let's check Example 18. If the area of the square is 9 square feet, the length of each side must be 3 feet. Namely, the only way we can get square feet as a denomination is by multiplying feet by feet. This tells us that the length of each side is expressed in feet. Secondly to find the area we multiply the length

of any side by itself. The only number that we can multiply by itself and get 9 as the product is 3. Hence the length of each side is 3 feet.

But if the square is 3 feet by 3 feet it also is 36 inches by 36 inches and this tells us that the area of the square is 36 X 36 or 1,296 square inches.

3 ft = 3 X 12 inches.

Note that if you had divided 1,296 by 12 you'd get 108 as the answer. If the area of the square was 108 square feet, then the length of each side would be more than 10 feet and hence more than 120 inches. But a 120 inch by 120 inch square has an area of 120 X 120 or 14,400 square inches which is much too big to be correct (since we're told that the area is 1,296 square inches)

A 10 foot by 10 foot square would have an area of 10 X 1 or 100 square feet. Hence if the area is 108 square feet, the length of each side must be longer than 10 feet.

Example 18 marks a change in emphasis of what we've been doing until now. Until now we have started with the length of the side of the square and found its area. But in analyzing Example 18 we started with the area of the square and computed the length of each side.

In terms of fill-in-the-blank, there are two ways of asking for the information that $36^2 = 1,296$. We can ask:

$$36^2 = \underline{\qquad} \qquad (1)$$

or we can ask

$$(\underline{\quad})^2 = 1,296 \qquad (2)$$

Compare this with addition and subtraction. To ask fc the fact that 3 + 2 = 5, we can write 3 + 2 = ___ (which is an addition proble or 3 + __ = 5 (which is a subtraction problem)

We invent a special notation for solving (2). The number whose square is 1,296 is called the

square root of 1,296 and is written as $\sqrt{1{,}296}$.

More generally:

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Definition

If n denotes any number, $\sqrt{n}$*,* *called the* square root of n, *denotes the number whose square in n. That is:*

$$\sqrt{n} \times \sqrt{n} = n$$

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$36^2 = 1{,}296$ and $\sqrt{1{,}296} = 36$ say the same thing. In one case we're saying that if the length of a side is 36 inches, then the area of the square is 1,296 square inches. In the other case we're saying that if the area of the square is 1,296 square inches then each side is 36 inches

In general it is more difficult to find the square root of a number than it is to square the number. In fact, if we deal with whole numbers notice that:

$0^2 = 0 \times 0 = 0$

$1^2 = 1 \times 1 = 1$

$2^2 = 2 \times 2 = 4$

$3^2 = 3 \times 3 = 9$

$4^2 = 4 \times 4 = 16$

$5^2 = 5 \times 5 = 25$

$6^2 = 6 \times 6 = 36$

$7^2 = 7 \times 7 = 49$

$8^2 = 8 \times 8 = 64$

$9^2 = 9 \times 9 = 81$

$10^2 = 10 \times 10 = 100$

The numbers 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, and so on are called perfect squares *because they're squares of whole numbers.*

Since any whole number ends in 0,1,2,3,4,5,6,7,8,or 9 a perfect square must end in 0,1,4,5,6,or 9. In particular, a perfect square cannot end in 2,3,7, or 8.

So any number between 0 and 100 except for 1, 4, 9, 16, 25, 36, 49, 64, and 81 is not a perfect square. That is, its square root is not a whole number.

Let's make sure that you understand the difference between squaring a number and taking the square root of a number.

This is similar to understanding the difference between knowing the sum of 5 and 8 and knowing what we must add to 5 to get 8 as the sum.

Example 19

Find the value of:

(a) 15^2 (b) 16^2 (c) 15.5^2

Answer: (a) 225 (b) 256 (c) 240.25

Squaring a number means tomultiply the number by itself. So we have:

(a) $15^2 = 15 \times 15 = 225$

(b) $16^2 = 16 \times 16 = 256$

(c) $15.5^2 = 15.5 \times 15.5 = 240.25$

If you have a calculator with a key labeled "x^2", you can find the square of a number by entering it on the calculator and then pressing the "x^2" key

Example 19 leads to a technique for computing square roots.

Example 20

Rounded off to the nearest whole number, what is the square root of 250? That is, how much is $\sqrt{250}$?

Answer: 16

We're looking for an answer to

$\_\_\_\_ \times \_\_\_ = 250$

In this case, the same number is used in each blank.

Since 15 X 15 = 225, 15 is too small to be correct; and since 16 X 16 = 256, 16 is too large to be correct. So the number we want is between 15 and 16.

From Example 19 (c) we know that 15.5 X 15.5 = 240.25 which is also too small to be correct. Hence, the square root of 250 is between 15.5 and 16. Hence to the nearest whole number it is 16.

$$15 \times 15 = 225$$
$$15.5 \times 15.5 = 240.25$$
$$\_\_\_\_ \times \_\_\_\_ = 250$$
$$16 \times 16 = 256$$

In terms of area if the area of a square is 250 sq in the length of each side is more than 15.5 inches but less than 16 inches.

Note

It is very tedious to find square roots to several decimal place accuracy. A very helpful device is to have a calculator with a key labeled "$\sqrt{x}$". If you have a calculator with this key simply enter the number whose square root you want and then press the "$\sqrt{x}$" key.

For example, on my calculator I enter "250". Then I press the square root key ($\sqrt{x}$) and almost immediately the display shows the decimal:

15.811388....

As a rough check:

$$\left.\begin{array}{l} 15.81^2 = 249.9561 \\ 15.82^2 = 250.2724 \end{array}\right\rangle 250$$

In concluding this discussion of square roots, let's revisit Example 20 in terms of areas.

Example 21

The area of a square is 250 square centimeters. To the nearest millimeter (that is, the nearest tenth of a centimeter) what is the length of each side of the square?

Answer: 15.8 cm or 158 mm

From Example 20 we already know that the length of each side is between 15 and 16 cm, but closer to 16 cm.

Make sure you have the label "15.8" and "158" are both wrong answers! It is crucial to distinguish between 158 mm and 158 cm.

If we use our above margin note, we know that to the nearest tenth of a centimeter, the length of each side is 15.8 cm.

See how we get the units? Since the area is in square centimeters, the length of each side is in centimeters.

Without a square root key, we would have to look at such products as:

$15.1^2 = 15.1 \times 15.1 = 228.01$

$15.2^2 = 15.2 \times 15.2 = 231.04$

$15.3^2 = 15.3 \times 15.3 = 234.09$

$15.4^2 = 15.4 \times 15.4 = 237.16$

$15.5^2 = 15.5 \times 15.5 = 240.25$

$15.6^2 = 15.6 \times 15.6 = 243.36$

$15.7^2 = 15.7 \times 15.7 = 246.49$

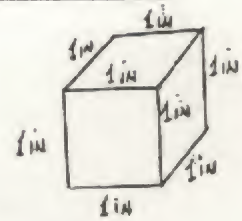
$$\left.\begin{array}{l} 15.8^2 = 15.8 \times 15.8 = 249.64 \\ 15.9^2 = 15.9 \times 15.9 = 252.81 \end{array}\right\rangle 250$$

There is a formula for finding a square root, but it is extremely cumbersome. I prefer that you either use an estimate or else use a calculator with a square root key.

Based on Example 19, we could have started with 15.6 X 15.6 because we already know that 15.5 X 15.5 is too small.

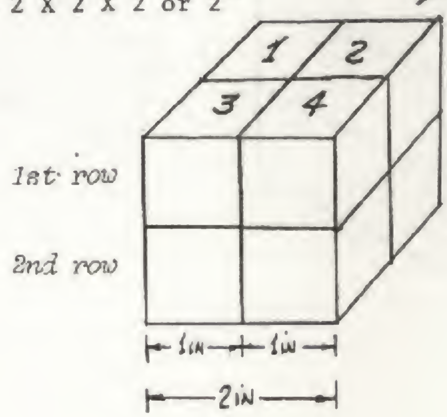
Just as we use squares to measure the amount of space inside a 2-dimensional region, we use *cubes* to measure the amount of space inside a 3-dimensional region.

A 1-inch cube:



As shown below, notice that there are *eight* 1-inch cubes inside one 2-inch cube. And it is no coincidence that $8 = 2 \times 2 \times 2$ or 2^3

2 X 2 = 4. So there are four 1" cubes in the 1st row and 4 more in the 2nd row. Hence there are eight 1"-cubes in one 2-inch cube.



In fact, based on this geometric fact we often reac 2^3 as "2 cubed" or "the cube of 2"

See what happens here? 2 X 2 tells us that there are 4 cubes in each layer; And since there are 2 layers we have 4 X 2 1-inch cubes in all.

****

To Find the Volume Of A Cube

(1) Take the length of each side including the label.

(2) Raise this to the 3rd power

****

Just as "area" refers to the amount of space in a 2-dimensional region, "volume" refers to the amount of space in a 3-dimensional region.

Example 22

The side of a cube is 4 cm long. What is the volume of the cube?

Answer: 64 cubic centimeters

The usual abbreviation for "cubic centimeters" is "cc"

Using the above "recipe" we have:

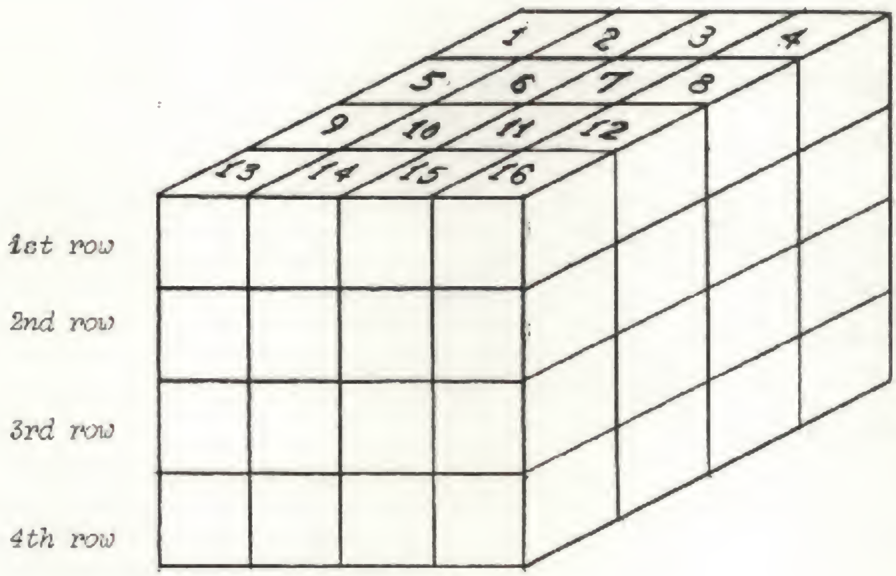
$4 \text{ cm} \times 4 \text{ cm} \times 4 \text{ cm} =$

$(4 \times 4 \times 4) \text{ cm} \times \text{cm} \times \text{cm} =$

$4^3 \text{ cm}^3 =$

64 cubic centimeters

In terms of a picture:



There are 4 X 4 or 16 1 cm-cubes per row and there are 4 rows. So we have 16 X 4 or 64 1-cm cubes in one 4-cm cube. In other words the 4-cm cube has a volume of 64 c.c. (cubic centimeters).

Additional remarks on area and volume appear in the Appendices at the end of this Module as well as in the Self-Test.

In concluding this Module we want to show you that there are many times, other than in areas and volumes, where we want to raise numbers to powers.

Category 3: Exponential Growth

How many times have we heard such remarks as the cost of living doubles every ten years? If this is true, there are some remarkable consequences. For example, suppose the cost of $1 object doubled every ten years. Let's go back to the year 1900 and see what the rate of growth of price looks like.

Year	Cost
1900	\$ 1
1910	\$ 2
1920	\$ 4
1930	\$ 8
1940	\$ 16
1950	\$ 32
1960	\$ 64
1970	\$128
1980	\$256

In 1910 the price is \$2, so this is the amount that's doubling between 1910 and 1920. So every ten years, the cost is double what it was ten years ago.

Example 23

Using the above chart, how much did the cost of the object increase by during the years between:

(a) 1900 and 1910? (b) 1930 and 1940?

Answer: (a) \$1 (b) \$8

Read the problem carefully. We want to know the increase in the price during the given time period. So we take the price at the end of the time period and *subtract* the price at the beginning of the time period. Hence:

If at the beginning of a we[ek] you have \$20 and if at the end of the week you have \$50, the increase is not \$5[0] but rather \$50 - \$20 or \$30

(a)
$$\begin{array}{rl} & \text{Cost in 1910} = \$2 \\ - & \underline{\text{Cost in 1900} = -1} \\ & \text{Increase} \quad = \$1 \end{array}$$

(b)
$$\begin{array}{rl} - & \text{Cost in 1940} = \$16 \\ & \underline{\text{Cost in 1930} = -\ 8} \\ & \text{Increase} \quad = \$\ 8 \end{array}$$

Since the cost is doubling, the increase is equal to th[e] cost at the beginning of th[e] time period.

So even though both parts (a) and (b) involved a ten-year time period, the change in the cost is different. That is, the rate of change of cost per ten-years is non-constant.

The relationship between the cost and the time period is called *exponential* because to find the present cost, we increase the exponent by 1 every ten years. That is:

$2^1 = 2$

$2^2 = 2 \times 2 = 4$

$2^3 = 2 \times 2 \times 2 = 8$

Each time we multiply the previous answer by 2. For example 2 X 2 X 2 X 2 is (2 X 2 X 2) X 2 or 8 X 2

$2^4 = 2 \times 2 \times 2 \times 2 = 16$

$2^5 = 16 \times 2 = 32$

$2^6 = 32 \times 2 = 64$

$2^7 = 64 \times 2 = 128$

The idea of the cost of living doubling every ten years can also be applied to how you save money. Most banks allow you to earn money by leaving your money with them. For example, if they tell you that your savings earn at an interest rate of 7% annually, it means that at the end of a year you have 107% of what you had in the bank. In other words, for each dollar you had in the bank, at the end of the year you have that dollar plus 7¢. So at the end of the year each dollar is worth $1.07.

It's like a 7% sales tax, except this time the money is paid to you.

<u>Example 24</u>

A bank pays an annual interest rate of 7%. How much interest will a deposit of $15,000 earn in one year?

Answer: $1,050

7% means that you earn 7¢ for each dollar you've deposited in the bank. Hence for $15,000 you earn 7¢, 15,000 times. So you've earned:

7¢ X 15,000 = 105,000 cents or $1,050.

Another way to say this is that you earn $7 for each $100; and there are 150 hundreds in 15,000. So you earn $7, 150 times.

The fact that the $15,000 earned $1,050 in interest means that you now have $16,050 on deposit.

100% = $15,000
7% = 1,050
107% = $16,050

Check: 1.07 X $15,000 = $16,050

To encourage you to leave your money in the bank, most banks agree to pay *compound* interest. For example referring to Example 24, during the second year the bank will pay you the 7% interest rate *on the entire $16,050.*

Example 25

Under the conditions of Example 24, how much interest will you earn **during the** second year?

Answer: $1,123.50

At the end of the first year (which is the same as the beginning of the second year), you have $16,050 on deposit. Hence the 7% rate applies to the entire $16,050. That is, during the second year, you earn

7% of $16,050 or

0.07 X $16,050 or

$1,123.50

Notice that the interest you earn each year is not constant. During the first year you earned $1,050 but during the second year you earned $1,123.50

Now watch how exponents are used in Examples 24 and 25. At the end of the first year, the total amount you had in the bank was 1.07 X $15,000 ($16,050). At the end of the second year yoy have 1.07 X (1.07 X $15,000) or 1.07 X $16,050 or $17,173.50. *But an easier way to write 1.07 X (1.07 X $15,000) is* 1.07^2 *X $15,000.*

Check:

$15,000	*(original amount)*
1,050	*(1st year's interest*
1,123.50	*(2nd year's int.)*
$17,173.50	

In other words, if the 7% interest rate is compounded annually, each year you'll have 1.07 times the amount you had the previous year.

Let's try a few more examples of this same type.

Example 26

A bank pays an interest rate of 7% compounded annually. If you deposit $15,000 how much money will you have in the bank at the end of 3 years? Round off the answer to the nearest cent.

Answer: $18,375.65

At the end of one year you'll have

1.07 X $15,000 or $16,050.

At the end of two years you'll have

1.07 X $16,050 or $17,173.50

At the end of three years you'll have

1.07 X $17,173.50 or $18,375.645, which

we round off to $18,375.65

In effect we took 107% of of 107% of 107% of $15,000. That is, we computed:

$1.07^3 \times \$15,000$

As you can begin to sense from Example 26, the arithmetic is not difficult but it is tedious; and gets more tedious as the number of years increase. For example, at the end of 10 years, you would have

$$1.07^{10} \times \$15,000$$

in the bank. But it is tedious to compute 1.07^{10}.

Some calculators come equipt with a "y^x" key. If you have such a key and you want to find 1.07^{10}, enter the 1.07, press the "y^x" key and then press the "=" key.

Before the era of calculators, interest tables were (and still are available

Of course, with a calculator you would just multiply 1.07 by 1.07, then multiply this answer by 1.07 and so on until you've taken the product of ten 1.07's.

You should also realize that the method we've used in the last few examples doesn't depend on the fact that we deposited $15,000. We're earning $0.07 per $1. So for $200 we're earning $0.07, 200 times while for a million dollars we're earning $0.07 a million times.

See? The rate is the same, but how many times we get the rate depends on how much we have in the bank.

If you were to do the computations, you'd find that at a rate of 7% compounded annually, a deposit would double in about 10 years. Hence it would double again the next ten years and so on.

So roughly speaking:

Year	Amount on Deposit (approximately)
1970	$15,000
1980	$30,000
1990	$60,000
2000	$120,000
2010	$240,000

and this is the principle behind most retirement plans. A small amount left on deposit today can be worth a small fortune at retirement.

But in terms of the message of this Module, look at the effect of the non constant-rate of growth. During the 10 years between 1970 and 1980 your investment increased by $15,000; but in the ten years between 2000 and 2010, your investment increased by $120,000.

The examples we've presented so far should be adequate for helping you become familiar with non-constant rates. In the next module, we shall show how the subject called *algebra* helps us in the study of rates of change.

$1.07^{10} = 1.9671514....$

$1.07^{11} = 2.1048519....$

So $1 becomes $2 in a little over 10 years. See? Rounded off to the nearest cent after 10 years a dollar becomes $1.97 and after 11 years it becomes $2.10. So during the 11th year the $1 doubles.

So if you deposited $15,000 in 1970 when you were 25 years old, in 2010 you'd be 65 years old and the $15,000 would have grown to $240,000

Of course the cost of living is increasing too. So it's possible that the $240,000 in 2010 might not be worth any more than $15,000 was in 1970. But at least you'd have the $240,000 if you saved the $15,000 in 1970

Appendix 1: More on Rectilinear Figures

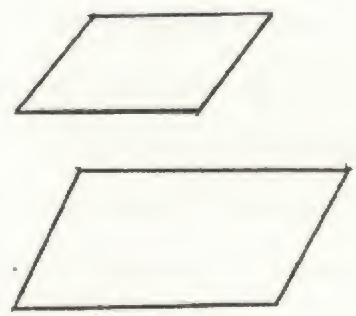
While it is easy to think about area, it isn't always easy to measure it. However for rectilinear regions (that is, for regions whose boundaries are made up of straight lines) there is a method that is relatively easy to use.

In dealing with rectilinear regions we don't spend much time discussing perimeter. Namely, since each side is a straight line, we can measure its length with a ruler; and then add the lengths of each side.

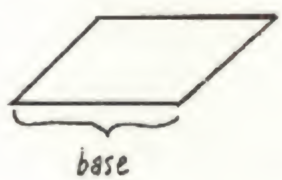
We start from the fact that we already know how to find the area of a rectangle. Once we know this we can then find the area of a *parallelogram*.

```
*********************************************
*   Definition:                             *
*                                           *
*   A parallelogram is a closed 4-sided     *
*   rectilinear region whose opposite       *
*   sides are parallel (i.e., have the      *
*   same direction.                         *
*********************************************
```

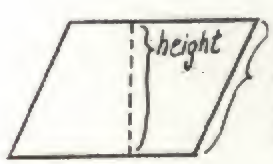
Some examples of parallelograms:



Now just as in the case of a rectangle, the base of a parallelogram is the side the parallelogram rests on. That is:

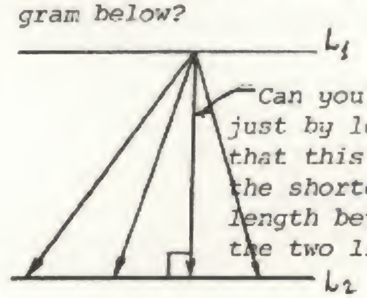


The height of a parallelogram is the distance between the base and its opposite side (the "top") That is:



Unless the figure is a rectangle, this side is NOT the height.

By the distance between two parallel lines, we mean the SHORTEST distance. This is the length that meets both lines at right angles. Can you see this from the diagram below?



Can you see just by looking that this is the shortest length between the two lines?

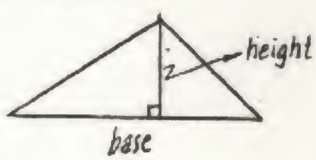
Appendix 1 (cont)

As described in the mrgin, the area of any parallelogram is the product of the base and the height (In this sense, then, a rectangle is just a special case of a parallelogram).

Once we can find the area of a parallelogram, we can find the area of any triangle.

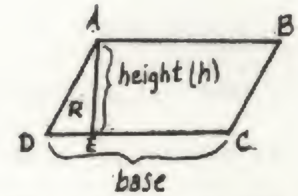
> *Definition:*
>
> *Any closed rectilinear figure consisting of three sides is called a TRIANGLE.*

Whichever side we assume the triangle rests on is called the base of the triangle. The line that is the shortest distance to the base from the opposite vertex is called the height of the triangle. That is:

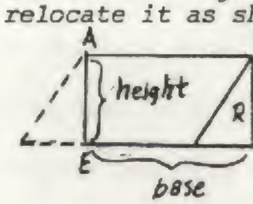


Then as shown in the margin, the area of any triangle is *half* the product of the base and the height.

Once we know how to find the area of a triangle we can then find the area of any rectilinear region simply by subdividing it into triangles. As example is shown on the next page. However at this level of mathematics it isn't too important to pursue this topic in more depth at this time.

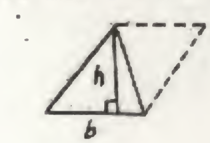


Cut out the region R and relocate it as shown below



The resulting figure is a rectangle that has the same area as the parallelogram (because we simply relocated a piece). Since the area of the rectangle is the product of the base and height, so also is the area of the parallelogram.

Now if we're given a triangle of base b and height h we can convert it into a parallelogram as indicated by the dotted lines below:

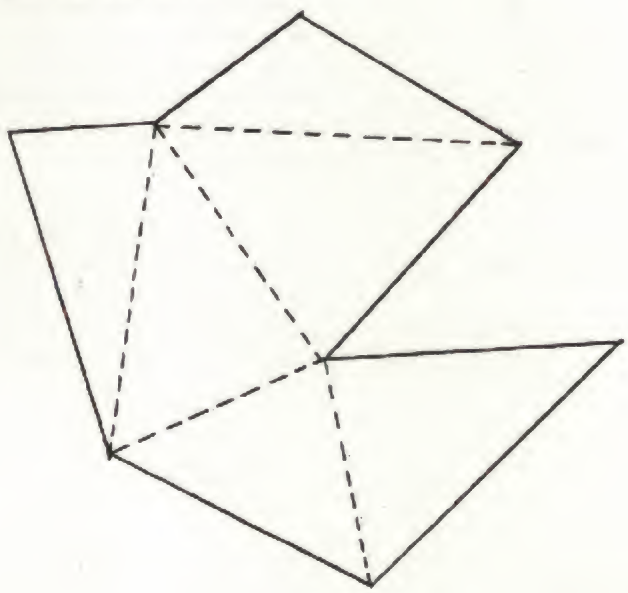


The triangle makes up half of the parallelogram. Since the area of the parallelogram is the product of the base and height (b X h), the are of the triangle is half the product of the base and height.

Appendix 1 (concluded)

Look at the rectilinear figure drawn below and look at the dotted lines. They divide the figure into triangles. So we can measure the area of each triangle (by measuring the base and height of each and taking the product of half the base and height). We can then add the areas of these triangles to get the area of the entire region.

This is a common method still used today by surveyors. The process is known as the method of triangulation.



Appendix 2: The Circle

Of all non-rectilinear figures, the circle is perhaps the most famous.

A special device called a compass is used to draw a circle.

****

Definition:

A circle is the set of all points whose distance from a fixed point is the same given distance. The fixed point is called the center *and the given distance is called the* radius.

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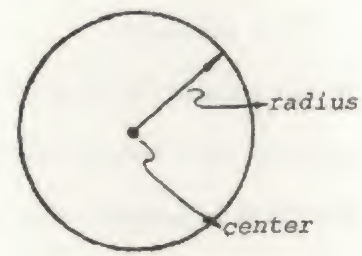
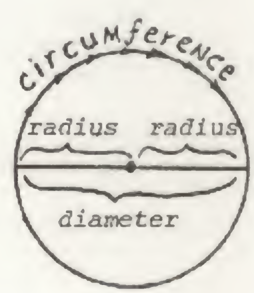


There is a nice way to visualize a circle in terms of area. Take a piece of string and tie the ends together. We can now let the string form a variety of shapes, each having the length of the string as a common perimeter. Even though the perimeters are the same, some figures contain more area then others. *Just as the square is the rectangle that encloses the greatest area, the circle is the figure that encloses the greatest area possible for a given perimeter.*

The reason the circle has this property is because of its perfect symmetry. In fact the shape of a circle is so unique that all circles have the same shape.

Think in terms of a wheel. No matter how much it changes position as it rolls, the shape of the circle never changes.



There are some special vocabulary words that are used when we talk about circles. First of all the perimeter of a circle is called its *circumference*. The length of a line segment that passes through the center of the circle

and which begins and ends on the circle is called a *diameter*.

The length of a diameter is twice the size of the radius. That is:

Because all circles have the same shape, the ratio between the circumference and the diameter of a circle is always the same. That is, if we denote the circumference of a circle by C and its diameter by D:

The bigger the radius, the bigger the circumference. But if we divide the circumference by the diameter, we always get the same quotient no matter what circle we pick

$$\frac{C}{D} = \text{constant}$$

This constant is denoted by the Greek letter π (spelled "pi" but pronounced "pie"). It is an irrational number, which to the nearest hundredth is 3.14 and is often estmated as being $3\frac{1}{7}$ or $\frac{22}{7}$.

Notice that since $C \div D = \pi$, we may also say:

$$C = \pi \times D \qquad (1)$$

and since the diameter is twice the radius, we may replace D by 2 X r to get:

$$C = \pi \times 2 \times r$$

As a rough estimate, the circumference of a circle is a bit more than 3 times the diameter and a bit more than 6 times the radius.

Hence if we know either the diameter or the circumference of a circle we can find the value of the other.

Example 27

If the diameter of a circle is 15" what is its circumference? Use 3.14 to stand for π.

Answer: 47.1"

All we have to do is use (1) with D = 15 (inches). We get:

$$C = 3.14 \times 15$$
$$= 47.1 \text{ (inches)}$$

Rough Check:

3 X 15 inches = 45 inches. So we expect an answer that's a bit more than 45 inches.

Example 28

Again using 3.14 for π, what is the diamter of a circle if its circumference is 15 inches? Write your answer to the nearest hundredth of an inch.

Answer: 4.78 inches

Rough Check:

15 ÷ 3 = 5, so we expect an answer of around 5 inches but a bit less since 3.14 is greater than 3.

We again use (1) with π = 3.14 but now its C that's replaced by 15. So we get:

$$15 = 3.14 \times D$$

and this tells us that

$$15 \div 3.14 \doteq 4.78$$

The actual quotient on my calculator is 4.7770701 but this isn't accurate because π isn't exactly 3.14. My calculator has a "π"key that indicates that π ≐ 3.1415927. Using this key I find that 15 ÷ π ≐ 4.7746483.... So perhaps the safest thing to say is that to the nearest tenth the answer is 4.8 inches.

Notice how easy it can be to confuse Examples 27 and 28 if you don't read formula (1) correctly.

A more complicated question concerns the area that's enclosed by a circle. The proof is beyond our scope but is discussed intuitively in Videotape Lecture 11B. In any event to find the area of a circle, we need only quare the radius and then multiply by π. That is, if we denote the area of the circle by A and its radius by r, we get:

$$A = \pi \times r^2 \qquad (2)$$

In words, to find the area of a circle, we:

(1) Start with the radius.

(2) Multiply it by itself.

(3) Multiply this product by π
The answer is the area of the circle

We can distinguish in part formulas (1) and (2) by recognizing that r^2 indicates an measure of area. That is, for example,
in X in = <u>square</u> inches

Appendix 2 (cont)

Example 29

Using 3.14 for π, what is the area of a circle if its radius is 6 cm?

Answer: 113.04 square cm

We start with 6 (cm), square it to get 36 (sq. cm.) and then multiply by 3.14 to get113.04 as the answer.

Some further dirll is left for the Self-Test, but otherwise this completes ~~our~~ introduction to the study of circles.